

A solution to the Pompeiu problem

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Abstract

Let $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}$, where \mathcal{S} is the Schwartz class of distributions, and

$$\int_{\sigma(D)} f(x) dx = 0 \quad \forall \sigma \in G, \quad (*)$$

where $D \subset \mathbb{R}^n$ is a bounded domain, the closure \bar{D} of which is diffeomorphic to a closed ball, and S is its boundary. Then the complement of \bar{D} is connected and path connected. By G the group of all rigid motions of \mathbb{R}^n is denoted. This group consists of all translations and rotations. A proof of the following theorem is given.

Theorem 1. *Assume that $n = 2$, $f \not\equiv 0$, and $(*)$ holds. Then D is a ball.*

Corollary. *If the problem $(\nabla^2 + k^2)u = 0$ in D , $u_N|_S = 0$, $u|_S = \text{const} \neq 0$ has a solution, then D is a ball.*

Here N is the outer unit normal to S .

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1 Introduction

Let $f \in L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}$, where \mathcal{S} is the Schwartz class of distributions, and

$$(1) \quad \int_{\sigma(D)} f(x) dx = 0 \quad \forall \sigma \in G,$$

where G is the group of all rigid motions of \mathbb{R}^n , G consists of all translations and rotations, and $D \subset \mathbb{R}^n$ is a bounded domain, the closure \bar{D} of which is diffeomorphic to a closed ball.

Under these assumptions the complement of \bar{D} in \mathbb{R}^n is connected and path connected ([5]). By S the boundary of D is denoted, and N denotes the unit normal to S pointing out of D . In [6] the following question was raised by D.Pompeiu:

Does (1) imply that $f = 0$?

If yes, then we say that D has P -property (Pompeiu's property), and write $D \in P$. Otherwise, we say that D fails to have P -property, and write $D \in \bar{P}$. Pompeiu claimed that every plane bounded domain has P -property, but a counterexample was given 15 years later in [2]. The counterexample is a domain D which is a disc, a ball in \mathbb{R}^n for $n > 2$. If D is a ball, then there are $f \not\equiv 0$ for which equation (1) holds. The set of all $f \not\equiv 0$, for which equation (1) holds, was constructed in [7]. A bibliography on the Pompeiu problem (P -problem) can be found in [14]. The results on P -problem which are used in this paper are derived in [12]. The P -problem is equivalent to a symmetry problem, see Corollaries 1,2 below. The author's results on other symmetry problems are given in [10] and [11]. The modern formulation of the P -problem is the following:

Prove that if $D \in \bar{P}$ then D is a ball.

We use the word ball also in the case $n = 2$, when this word means disc, and solve the P -problem. The proof of Theorem 1 we give assuming $n = 2$, but this proof is easily generalized to the case $n > 2$. Our standing assumptions are:

Assumptions A: a) D is a bounded domain, the closure of which is diffeomorphic to a closed ball, the boundary S of D is a closed connected C^1 -smooth surface, b) D fails to have P -property, and c) $n = 2$.

Theorem 1. *If Assumptions A hold, then D is a ball.*

Corollary 1. *If problem (3) (see below) has a solution, then D is a ball.*

Corollary 2. *If the problem $(\nabla^2 + k^2)u = 0$ in D , $u_N|_S = 0$, $u|_S = \text{const} \neq 0$ has a solution, then D is a ball.*

In Section 2 these results are proved.

2 Proof of Theorem 1

If Assumptions A hold, then the boundary S of D is real-analytic (see [13]) and

$$(2) \quad \int_D e^{ik\alpha \cdot x} dx = 0, \quad \forall \alpha \in S^1,$$

where S^1 is the unit sphere in \mathbb{R}^2 , and $k > 0$ is a fixed number, see [12].

The following Lemmas 1-3 are proved in [12] (Lemma 1 is Lemma 3 in [12], Lemma 2 is Lemma 5 in [12], and Lemma 3 is formula (32) in [12]):

Lemma 1. *If and only if relation (2) holds then the overdetermined problem*

$$(3) \quad (\nabla^2 + k^2)u = 1 \quad \text{in } D, \quad u|_S = 0, \quad u_N|_S = 0,$$

has a solution.

Lemma 2. *If (2) holds for all $\alpha \in S^1$ then it holds for all $\alpha \in M$, where $M := \{z : z \in \mathbb{C}^2, z_1^2 + z_2^2 = 1\}$.*

The M is an algebraic variety intersecting \mathbb{R}^2 over S^1 .

Let us assume that the boundary S is star-shaped. Let $r = f(\phi)$ be the equation of S , where $0 < c_2 \leq f \leq c_1$, c_j are constants, $j = 1, 2$, and f is a smooth 2π -periodic function.

Lemma 3. *If (2) holds for all $\alpha \in S^1$, then*

$$(4) \quad \int_{-\pi}^{\pi} f'(\phi) f(\phi) e^{ikf(\phi) \cos(\phi-\theta)} d\phi = 0, \quad \forall \theta \in \mathbb{C}.$$

Let us choose $\cos \theta = is$ and $\sin \theta = (s^2 + 1)^{1/2}$. Then $\{is, (s^2 + 1)^{1/2}\} \in M$, and (4) can be written as

$$(5) \quad \int_{-\pi}^{\pi} f'(\phi) f(\phi) e^{-skf(\phi) \cos \phi + ik(s^2+1)^{1/2} f(\phi) \sin \phi} d\phi = 0, \quad \forall s > 0.$$

Multiply (5) by e^{-As} , where $A > 0$ is a large constant, and integrate over s from 0 to ∞ . Then one gets

$$(6) \quad \int_{-\pi}^{\pi} d\phi f'(\phi) f(\phi) \int_0^{\infty} ds e^{-s(a+A) + i(s^2+1)^{1/2} b} = 0, \quad \forall A > A_0,$$

where $A_0 > 0$ is a fixed large constant,

$$a = a(\phi) = kf(\phi) \cos \phi, \quad b = b(\phi) = kf(\phi) \sin \phi, \quad A_0 > \max_{\phi \in [-\pi, \pi]} |a(\phi)|.$$

One has

$$(7) \quad \int_0^{\infty} e^{-s(a+A) + i(s^2+1)^{1/2} b} ds = (a+A)^{-1} e^{ib} [1 + O(A^{-1})] ds, \quad A \rightarrow \infty.$$

Writing

$$(a+A)^{-1} = \sum_{j=0}^{\infty} (-1)^j a^j A^{-1-j}, \quad A > A_0,$$

one obtains from (6) and (7) the relation

$$(8) \quad \int_{-\pi}^{\pi} f'(\phi) f(\phi) e^{ib} \sum_{j=0}^{\infty} (-1)^j a^j A^{-1-j} [1 + O(A^{-1})] d\phi = 0, \quad A \rightarrow \infty.$$

Multiply (8) by A and let $A \rightarrow \infty$. This yields relation (9), see below, with $j = 0$. After getting relation (9) with $j = 0$, multiply (8) by A^2 and let $A \rightarrow \infty$. This yields relation (9) with $j = 1$. Continue in this fashion to get

$$(9) \quad \int_{-\pi}^{\pi} f'(\phi) f(\phi) a^j e^{ib} d\phi = 0, \quad \forall j = 0, 1, \dots$$

Applying the Laplace method (see [3]) for calculating the asymptotic behavior of integral (9) as $j \rightarrow \infty$, one concludes that (9) can hold if and only if $f' = 0$, that is, if and only if $f = \text{const}$.

Let us give details. Consider the function $a^{2m} = e^{m\Psi}$, where $\Psi := \ln[k^2 f^2(\phi) \cos^2 \phi]$, $j = 2m$, so that the expression under the logarithm sign is non-negative. The stationary points of the function Ψ are found from the equation $\frac{f'(\phi)}{f(\phi)} - \tan \phi = 0$.

If D is not a ball, then the function $f(\phi)$ attains its maximum F at a point, which one may denote $\phi = 0$. There can be finitely many points at which f attains local maximums, because f is analytic. There are finitely many points at which f attains the value F . We assume for simplicity that these points are non-degenerate, so $f'' < 0$ at these points. Since $f > 0$, one has the inequality

$$\frac{d}{d\phi} \left(\frac{f'(\phi)}{f(\phi)} - \tan \phi \right) = \frac{f''(\phi)}{f(\phi)} - \frac{(f'(\phi))^2}{f^2(\phi)} - \frac{1}{\cos^2 \phi} < 0,$$

if $f'' < 0$. Therefore, the critical points are non-degenerate and the main term of the asymptotic of the integral (9) with $j = 2m$ as $m \rightarrow \infty$, corresponding to the stationary point $\phi = 0$ can be calculated as follows. Let I denote the integral in (9). The stationary point $\phi = 0$ is a non-degenerate interior point of maximum of f and, therefore, of Ψ . Since $e^{ib(\phi)} = 1 + ikf(\phi) \sin \phi + \dots$, $f(\phi) = f(0) + f'(0)\phi + f''(0)\phi^2/2 + \dots$ and $f'(\phi) = f'(0) + f''(0)\phi + f'''(0)\phi^2/2 + \dots$, $\Psi(\phi) = \Psi(0) - \gamma\phi^2 + \dots$, where $\gamma := |\Psi''(0)|$, one multiplies the three terms $f f' e^{ib}$, takes into account that $f'(0) = 0$ and $\Psi'(0) = 0$ at the critical point, and gets $I \sim e^{m\Psi(0)} J$, where

$$J = \int_{[-\delta, \delta]} \left((ikf^2(0)f''(0) + f(0)f'''(0)/2)\phi^2 + \dots \right) e^{-m\gamma\phi^2} d\phi.$$

As $m \rightarrow \infty$, one extends the interval of integration to $(-\infty, \infty)$ and calculates the main term of the asymptotic of J as $m \rightarrow \infty$ by using the formula

$$\int_{-\infty}^{\infty} \phi^2 e^{-m\gamma\phi^2} d\phi = \frac{\Gamma(3/2)}{(m\gamma)^{3/2}},$$

where $\Gamma(z)$ is the Gamma-function, $\Gamma(3/2) = \sqrt{\pi}/2$. The result is

$$(10) \quad I \sim e^{m\Psi(0)} \frac{\Gamma(3/2)}{(m\gamma)^{3/2}} \left(ik f^2(0) f''(0) + f(0) f'''(0)/2 \right), \quad m \rightarrow \infty.$$

Since $I = 0$ and $f(0) > 0$, one concludes from (10), after taking the imaginary part, that $f''(0) = 0$, and after taking the real part, that $f'''(0) = 0$. This contradicts the non-degeneracy of the critical point $\phi = 0$. If one does not assume the non-degeneracy of this critical point, then one uses the analyticity of the function f and concludes that if for some j the derivative $f^{(j)}(0) \neq 0$, then this leads to a contradiction. Thus, all the derivatives $f^{(j)}(0) = 0$ for $j > 0$. Each critical point at which $f = F$ can be taken to be the point $\phi = 0$, because the origin for ϕ in formula (4) can be chosen arbitrarily on the interval of length of the period $[0, 2\pi]$.

If the critical point $\phi = 0$ is non-degenerate then the inputs of local maximums at which $f = F$ cannot compensate each other since their imaginary parts are all of the same sign since $f'' < 0$ and $f > 0$ at these points. There can be at most finitely many critical points of f since f is analytic.

Thus, the only possibility to have equalities (9) for all large j is to have $f = \text{const}$.

Theorem 1 is proved. \square

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